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Journal of Applied Mathematics and Mechanics 71 (2007) 30–39
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

A problem of the bending of a plate for a doubly connected domain with a partially unknown boundary☆

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Received 14 June 2005

Abstract

The problem of the bending of an isotropic elastic plate, bounded by two rectangles with vertices lying on the same half-line, drawn from the common centre, is considered. The vertices of the inner rectangle are cut by convex smooth arcs (we will call the set of these arcs the unknown part of the boundary). It is assumed that normal bending moments act on each rectilinear section of the boundary contours in such a way that the angle of rotation of the midsurface of the plate is a piecewise-constant function. The unknown part of the boundary is free from external forces. The problem consists of determining the bending of the midsurface of the plate and the analytic form of the unknown part of the boundary when the tangential normal moment acting on it takes a constant value, while the shearing force and the normal bending moments and torques are equal to zero. The problem is solved by the methods of the theory of boundary-value problems of analytical functions.

Similar problems of the bending of a plate and of the plane theory of elasticity were investigated previously for an infinite plate, weakened by apertures with unknown equal strength contours,[1–5] and for a finite doubly connected region with a partially unknown boundary.[6,7] Unlike those problems, here we consider the case when the required contour consists of separate smooth arcs.

1. Formulation of the problem

We will denote by S the region occupied by the midsurface of the plate, and by $L_0^{(k)} = A_k A_{k+1} (k = 1, \ldots, 4\ A_5 \equiv A_1)$, $L_1^{(j)} = B_{2j-1} B_{2j} (j = 1, \ldots, 4)$ and $L_2^{(m)} (m = 1, \ldots, 4)$ sections and smooth arcs of the outer and inner boundaries of the region S respectively, and we will assume that

$$L_j = L_j^{(1)} \cup L_j^{(2)} \cup L_j^{(3)} \cup L_j^{(4)}, \quad j = 0, 1, 2$$

We will also assume that the sections $L_j^{(2k-1)} (j = 0, 1; k = 1, 2)$ are parallel to the ordinate axis, while the sections $L_j^{(2k)} (j = 0, 1; k = 1, 2)$ are parallel to the abscissa axis (Fig. 1), and we will choose as the positive direction on the boundary $L = L_0 \cup L_1 \cup L_2$ that which leaves the region S on the left.

☆ *Prikl. Mat. Mekh.* Vol. 71, No. 1, pp. 33–42, 2007.
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0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2007.03.011

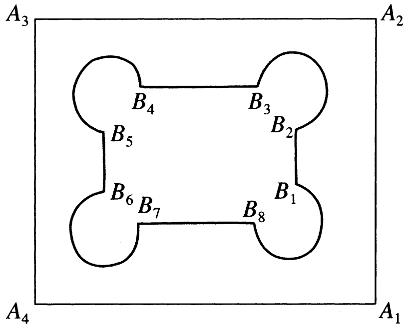

Fig. 1.

We will assume that normal bending moments act on each rectilinear section of the boundary contours and hence that the angles of rotation of the midsurface of the plate take piecewise-constant values, while the unknown part of the boundary is free from external forces.

We will consider the following problem: it is required to obtain the bending of the midsurface of the plate and the analytical form of the unknown part of the boundary such that the tangential normal moment $M_s(t)$ acting on it takes a constant value (k_0), while the shearing force, the normal bending moments and torques are equal to zero.

2. Solution of the problem

According to the approximate theory of the bending of a plate,[8–12] the sag $W(x, y)$ of the midsurface of the plate in the case considered satisfies the biharmonic equation

$$\Delta^2 W(x, y) = 0, \quad z = x + iy \in S \tag{2.1}$$

and the boundary conditions

$$\begin{aligned} &M_n(t) = f(t) \text{ (or } \partial W/\partial n = d(t)); \quad N(t) = 0, \quad t \in L_0 \cup L_1 \\ &M_n(t) = 0; \quad M_{ns}(t) = 0; \quad M_s(t) = k_0 = \text{const}; \quad N(t) = 0, \quad t \in L_2 \end{aligned} \tag{2.2}$$

where $d(t) = d_k = tg\ \lambda_k$ (γ_k are the angles of rotation), $t \in L_0^{(k)} \cup L_1^{(k)}$, $N(t)$ is the shearing force, and $M_n(t)$ and $M_{ns}(t)$ are the normal bending moments and torques respectively. Note that the boundary conditions on the contour L_2 are “overdetermined” (in the classical formulations of the problem we have two conditions on the free part of the boundary: $M_n(t) = 0$, $N(t) + \partial M_{ns}/\partial s = 0$), but in the case of an equal strength contour, characterized by the condition $M_s(t) = k_0 = \text{const}$, $t \in L_2$, as will be seen below, the condition $N(t) = 0$ is satisfied automatically, and, hence, on the contour L_2 two conditions remain

$$M_n(t) = 0, \quad M_{ns}(t) = 0, \quad t \in L_2$$

Problem (2.1), (2.2) is solved using a scheme similar to that described in Ref. 13.

We will assume that the set of all the external forces applied along the contour L_1 are statically equivalent to zero. In this case, the problem reduces to finding two functions $\varphi(z)$ and $\psi(z)$, holomorphic in the region S, with boundary conditions on the contour L

$$\text{Re}\{e^{-i\nu(t)}[\varphi(t) + i\overline{\varphi'(t)} + \overline{\psi(t)}]\} = d_j(t)\ t \in L_j \tag{2.3}$$

$$\text{Re}\{e^{-i\nu(t)}[\kappa\varphi(t) - \overline{t\varphi'(t)} - \overline{\psi(t)}]\} = F_j^{(k)}(t)\ t \in L_j^{(k)}, \quad j = 0, 1; \quad k = 1, \ldots, 4 \tag{2.4}$$

$$\kappa\varphi(t) - \overline{t\varphi'(t)} - \overline{\psi(t)} = E(t), \quad t \in L_2 \tag{2.5}$$

$$\text{Re}[\varphi'(t)] = p, \quad t \in L_2 \tag{2.6}$$

where

$$\nu(t) = \alpha(t), \quad t \in L_0; \quad \nu(t) = \beta(t), \quad t \in L_1$$

$$F_j^{(k)}(t) = [2D^0(\sigma-1)]^{-1} \sum_{m=1}^{k} \int_{L_j^{(m)}} M_n(t)\sin(\nu_k - \nu_m)ds + c_j^{(k)}$$

$$p = -k_0[8D^0(\sigma+1)]^{-1}, \quad \kappa = (\sigma+3)(1-\sigma)^{-1}$$

$\alpha(t)$ and $\beta(t)$ are the angles between the x axis and the outward normals of the contours L_0 and L_1 at the point $t \in L_0 \cup L_1$, $c_j^{(k)}$ are arbitrary real constants, and $E(t) = E_k$, $t \in L_2^{(k)}$ ($j = 0, 1; k = 1, \ldots, 4$) are arbitrary (generally speaking, complex) constants (D^0 is the cylindrical stiffness of the plate and σ is Poisson's ratio).

We will require the function $\varphi(z)$ to be continuous in the closed region $S+L$, while the functions $\varphi'(z)$ and $\psi(z)$ are required to be continuously extendable on the boundary of the region S everywhere with the possible exception of the points A_k and B_k, in the neighbourhood of which they satisfy the condition

$$|\varphi'(z)|, |\psi(z)| < M|z-c|^{-\delta_0}; \quad M = \text{const}, \quad 0 \le \delta_0 \le 1/2 \tag{2.7}$$

Suppose the function $z = \omega(\zeta)$ conformally maps the circular ring $D(1 < |\zeta| < R)$ onto the region S. We will assume that the circle $(l_0(|\zeta|) = R)$ is mapped on to the contour L_0, while the circle $(l_{00}(|\zeta| = 1)$ is mapped on to the contour $L_1 \cup L_2$. We will denote by $a_k(k = 1, \ldots, 4)$ and $b_k(k = 1, \ldots, 8)$ the originals of the points A_k and B_k, and by l_1 and l_2 the parts of the circle l_{00}, corresponding to the lines L_1 and L_2, and we will assume that

$$l_j = l_j^{(1)} \cup l_j^{(2)} \cup l_j^{(3)} \cup l_j^{(4)}, \quad j = 0, 1, 2$$

where $l_j^{(k)}$ are arcs of the circle l_0 and l_{00}, corresponding to the contours $L_j^{(k)}$. Adding equalities (2.3) and (2.4) and then differentiating with respect to s, taking equality (2.6) into account, we obtain a mixed-type Dirichlet boundary-value problem with respect to the function $\Phi(\zeta) = \varphi'[\omega(\zeta)] - p$ for the circular ring D

$$\text{Re}\Phi(t) = 0, \quad t \in l_2; \quad \text{Im}\Phi(t) = 0, \quad t \in l_0 \cup l_1 \tag{2.8}$$

By virtue of conditions (2.7) it can be proved that problem (2.8) only has a trivial solution and, consequently, we have

$$\varphi(z) = pz \tag{2.9}$$

(we will assume that the arbitrary constant of integration is equal to zero). We conclude from the second formula that $N(z) = 0$, $z \in S+L$, and hence the condition $N(t) = 0$, $t \in L$ is satisfied automatically. On the basis of relations (2.4), (2.5) and (2.9) with respect to the functions $\omega(\zeta)$ and $\psi_0(\zeta) = \psi[\omega(\zeta)]$, holomorphic in the ring D, we obtain the boundary-value problem

$$\begin{aligned} &\text{Re}\{e^{-i\nu(\sigma)}[p(\kappa-1)\omega(\sigma) - \overline{\psi_0(\sigma)}]\} = F_j^{(k)}(\sigma), \quad \sigma \in l_j^{(k)}; \quad j = 0, 1; \quad k = 1, \ldots, 4 \\ &p(\kappa-1)\omega(\sigma) - \overline{\psi_0(\sigma)} = E(\sigma), \quad \sigma \in l_2 \end{aligned} \tag{2.10}$$

(to simplify the notation, the piecewise-constant functions $\nu[\omega(\sigma)], \ldots$ are again denoted by $\nu(\sigma), \ldots$; we will act in the same way in what follows with respect to the piecewise-constant functions and we will define them over the whole plane by the equalities $\nu(r\sigma) = \nu(\sigma), \ldots, 0 < r < \infty$.

It is easily seen that on the contours l_0 and l_1 we have the equalities

$$\text{Re}[e^{-i\nu(\sigma)}\omega(\sigma)] = f_j(\sigma), \quad \sigma \in l_j, \quad j = 0, 1 \tag{2.11}$$

where

$$f_j(\sigma) = \text{Re}[e^{-i\nu(\sigma)}A(\sigma)]; \ A(\sigma) = A_k, \ \sigma \in l_0^{(k)}; \ A(\sigma) = B_{2k-1}, \ \sigma \in l_1^{(k)}; \ k = 1, \ldots, 4$$

(A_k and B_{2k-1} are affixes of the points A_k and B_{2k-1}). Consider the function

$$W(\zeta) = \begin{cases} p(\kappa-1)\omega(\zeta), & 1<|\zeta|<R \\ \psi_{0*}(\zeta), & R^{-1}<|\zeta|<1, \quad \psi_{0*}(\zeta) = \overline{\psi_0(1/\bar{\zeta})} \end{cases} \tag{2.12}$$

From boundary conditions (2.10) and (2.11) we obtain a boundary-value problem of the theory of analytic functions of a mixed type in the functions $W(\zeta)$ for a circular ring D^* ($R^{-1}<|\zeta|<R$), cut along the arc of the circle l_{00}:

$$W(\sigma)+k^0(\sigma)\overline{W(\sigma)} = g(\sigma), \quad \sigma \in l_0 \cup l_0^* \tag{2.13}$$

$$W^{\pm}(\sigma)+k^1(\sigma)\overline{W^{\pm}(\sigma)} = g^{\pm}(\sigma), \quad \sigma\in l_1; \quad W^+(\sigma)-W^-(\sigma) = E(\sigma), \quad \sigma\in l_2 \tag{2.14}$$

where

$$k^j(\sigma) = (-1)^{m-1}, \quad \sigma\in l_j^{(m)}; \quad j = 0,1; \quad m = 1,\dots,4$$

$g(\sigma)$ and $g^{\pm}(\sigma)$ are functions, expressions for which can easily be written out (this will be done below in final form), and l_0^* is the transform of the circle l_0 for the mapping $\zeta_1 = \zeta/R^2$.

Consider the piecewise-holomorphic functions

$$\Omega_{1,2}(\zeta) = [W(\zeta)\pm W_*(\zeta)]/2$$

They satisfy the condition

$$\Omega_j(\zeta) = \Omega_{j*}(\zeta), \quad j = 1,2 \tag{2.15}$$

while the function $W(\zeta)$ is defined in terms of them by the formula

$$W(\zeta) = \Omega_1(\zeta)-i\Omega_2(\zeta) \tag{2.16}$$

Substituting expressions (2.16) into Eq. (2.14), we obtain the following boundary-value problems for the functions $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$

$$\begin{aligned} \Omega_j^+(t)+\Omega_j^-(t) &= h_j^{(1)}(t), \quad t\in \bigcup_{k=1}^{2} l_1^{(2k+j-2)} \\ \Omega_j^+(t)-\Omega_j^-(t) &= h_j^{(2)}(t), \quad t\in \bigcup_{k=1}^{2} l_1^{(2k+j-1)}\cup l_2 \end{aligned} \tag{2.17}$$

for $j=1$ and $j=2$ respectively, where $h_j^{(k)}(t)(j,k=1,2)$ are functions, expressions for which can easily be written down.

We will seek solutions of problems (2.17) of the class $h(b_1,\dots,b_8)$ (regarding this class see Ref. 8). The indices of these problems of the given class are equal to -2, while the solutions, which satisfy condition (2.15), have the form

$$\Omega_j(\zeta) = R_j(\zeta)I_j(\zeta)+R_j(\zeta)\omega_j(\zeta), \quad j = 1,2 \tag{2.18}$$

where

$$I_j(\zeta) = \frac{1}{2\pi i}\left[\int_{l_{00}}\frac{h_j(t)dt}{R_j(t)(t-\zeta)}-\frac{1}{2}\int_{l_{00}}\frac{h_j(t)dt}{tR_j(t)}\right]$$

$$R_j(\zeta) = R_j^0(\zeta)\prod_{k=0}^{1}(b_{4k+2j-1}b_{4k+2j})^{-1/4}[(\zeta-b_{4k+2j-1})(\zeta-b_{4k+2j})]^{1/2}, \quad j = 1,2$$

$$R_1^0(\zeta) = i/(\zeta^2-1), \quad R_2^0(\zeta) = i/(\zeta^2+1)$$

(the functions $h_j(t)$ are expressed in terms of the function $h_j^{(k)}(t)(j, k = 1, 2)$, and $\omega_j(\zeta)$ are arbitrary functions, holomorphic in the ring D^*, which satisfy the condition $\omega_j(\zeta) = \omega_{j^*}(\zeta)$, $(j = 1, 2)$.

It follows from the condition for the function $W(\zeta)$ to be bounded in the ring D^* that, among the constants $c_1^{(k)}$ and E_k $(k = 1, \ldots, 4)$, four remain arbitrary (we will assume that these are the constants $c_1^{(k)}$), while the remaining ones are expressed in terms of these. By virtue of relations (2.16) and (2.18), we obtain from the conditions (2.13) a Riemann-Hilbert boundary-value problem in the functions $\omega_j(\zeta)$ $(j = 1, 2)$ for the circular ring D^*

$$\omega_j(t) + G_j(t)\overline{\omega_j(t)} = g_j^{(k)}(t)R_j(t), \quad t \in l_0^{(k)} \cup l_0^{*(k)}, \quad j, k = 1, 2 \tag{2.19}$$

where

$$G_j(t) = (-1)^{k+j-2}\overline{R_j(t)}[R_j(t)]^{-1}$$

and $g_j^{(k)}(t)$ are certain functions which satisfy the condition

$$g_j^{(k)}(t) = g_j^{(k)}(t/R^2), \quad t \in l_0$$

If we take into account the fact that the indices of the functions $G_j(t)$ $(j = 1, 2)$ both on the contour l_0 and on the contour l_0^* are equal to zero, these functions can be represented in the form

$$G_j(t) = H_j(t)[\overline{H_j(t)}]^{-1}, \quad t \in l_0 \cup l_0^*, \quad j = 1, 2 \tag{2.20}$$

where $H_j(\zeta)$ $(j = 1, 2)$ is the solution of the Dirichlet problem for the ring D^*, i.e.

$$\ln H_j(t) - \ln\overline{H_j(t)} = \ln G_j(t), \quad t \in l_0 \cup l_0^*, \quad j = 1, 2 \tag{2.21}$$

The necessary and sufficient conditions for problem (2.21) to be solvable have the form

$$\prod_{k=0}^{1} b_{4k+2j-1}b_{4k+2j} = 1, \quad j = 1, 2 \tag{2.22}$$

while the solutions themselves can be represented in the form

$$H_j(\zeta) = H_0^{(j)}(\zeta)T_j^*(\zeta), \quad j = 1, 2 \tag{2.23}$$

where

$$H_0^{(j)}(\zeta) = \prod_{k=0}^{1} F(R^2; \zeta/b_{4k+2j-1}; 1/2)F(R^2; \zeta/b_{4k+2j}; 1/2) \times$$

$$\times \prod_{n=1}^{\infty}\prod_{k=0}^{1} F(R^{4n+2}; \zeta/b_{4k+2j-1}; 1/2)F(R^{4n}; \zeta/b_{4k+2j-1}; 1/2) \times$$

$$\times F(R^{4n+2}; \zeta/b_{4k+2j}; 1/2)F(R^{4n}; \zeta/b_{4k+2j}; 1/2)$$

$$T_j^*(\zeta) = F(R^4; (-1)^{j-1}\zeta^2; -1)\prod_{n=1}^{\infty} F(R^{8n}; (-1)^{j-1}\zeta^2; -1)F(R^{8n+4}; (-1)^{j-1}\zeta^2; -1)$$

$$j = 1, 2$$

$$F(\rho, \eta, \alpha) = \left[\left(1 - \frac{\eta}{\rho}\right)\left(1 - \frac{1}{\rho\eta}\right)\right]^{\alpha}$$

The functions $H_j(\zeta)$ $(j = 1, 2)$ satisfy the condition $H_j(\zeta) = H_{j^*}(\zeta)$, $\zeta \in D^*$.

We will now consider boundary-value problems of a mixed type for the ring D^*

$$\chi_j(t)+(-1)^{k+j}\overline{\chi_j(t)} = 0, \quad t\in l_0^{(k)}\cup l_0^{*(k)}; \quad j = 1,2; \quad j = 1,\ldots,4 \tag{2.24}$$

We will seek solutions of problems (2.24) of the class $h(a_1, \ldots, a_4)$ (the indices of these problems of the given class are equal to -4). The necessary and sufficient condition for problem (2.24) to be solvable has the form

$$\prod_{k=1}^{4} a_k = R^4 \tag{2.25}$$

while the solutions themselves have the form

$$\chi_j(\zeta) = \chi_a(\zeta)T_j(\zeta), \quad j = 1,2 \tag{2.26}$$

where

$$\chi_a(\zeta) = \prod_{k=1}^{4} F(R^2;\zeta/a_k;1/2)\prod_{n=1}^{\infty}\prod_{k=1}^{4} F(R^{4n};\zeta/a_k;1/2)F(R^{4n};R^2\zeta/a_k;1/2)$$

$$T_j(\zeta) = F(R^2;(-1)^{j-1}\zeta^2;-1)\prod_{n=1}^{\infty} F(R^{8n};(-1)^{j-1}\zeta^2/R^2;-1)F(R^{8n};(-1)^{j-1}\zeta^2R^2;-1)$$

$$j = 1,2$$

The functions $\chi_j(\zeta)$ $(j=1,2)$ satisfy the condition $\chi_j(\zeta)=\chi_{j^*}(\zeta)$. On the basis of these results, boundary conditions (2.19) can be written in the form

$$\Theta_j(t)-\overline{\Theta_j(t)} = \Gamma_j(t), \quad t\in l_0\cup l_0^*, \quad j = 1,2 \tag{2.27}$$

where

$$\begin{aligned}\Theta_j(\zeta) &= \omega_j(\zeta)[H_j(\zeta)\chi_j(\zeta)]^{-1}, \quad \zeta\in D^*\\ \Gamma_j(t) &= g_j^{(k)}(t)[H_j(t)\chi_j(t)R_j(t)]^{-1}, \quad t\in l_0^{(k)}\cup l_0^{*(k)}; \quad j = 1,2; \quad k = 1,\ldots,4\end{aligned} \tag{2.28}$$

The functions $\Gamma_j(t)$ satisfy the condition

$$\Gamma_j(t) = -\overline{\Gamma_j(t)} = -\Gamma_j(t/R^2), \quad t\in l_0, \quad j = 1,2$$

The necessary and sufficient conditions for problem (2.27) to be solvable have the form

$$\int_{l_0}\frac{\Gamma_j t}{t}dt = 0, \quad j = 1,2 \tag{2.29}$$

while the solutions themselves are represented by the formula

$$\Theta_j(\zeta) = \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\int_{l_0}\left[\frac{1}{t-R^{4n}\zeta}+\frac{1}{t-R^{4n}R^2\zeta}\right]\Gamma_j(t)dt+k_j, \quad j = 1,2 \tag{2.30}$$

where $k_j (j=1, 2)$ are real arbitrary constants.

The functions $\Theta_j(\zeta)$ $(j=1, 2)$ satisfy the condition $\Theta_j(\zeta)=\Theta_{j^*}(\zeta)$.

We can conclude on the basis of formula (2.28) that the functions $\Theta_j(\zeta)$ have the form

$$\omega_j(\zeta) = H_j(\zeta)\chi_j(\zeta)\Theta_j(\zeta), \quad \zeta\in D^*, \quad j = 1,2 \tag{2.31}$$

Hence, the functions $\Omega_j(\zeta)$ $(j=1, 2)$ can be represented in the form

$$\Omega_j(\zeta) = R_j(\zeta)I_j(\zeta) + R_j(\zeta)H_j(\zeta)\chi_j(\zeta)\Theta_j(\zeta), \quad \zeta \in D^*, \quad j = 1,2 \tag{2.32}$$

We can conclude on the basis of this formula that the functions $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$ have poles of the first order respectively at the points $\zeta = \pm 1, \pm R, \pm 1/R$ and $\zeta = \pm i, \pm iR, \pm i/R$. In order for these functions to be holomorphic in the ring D^*, the following conditions must be satisfied

$$I_1(\pm 1) + H_1(\pm 1)\chi_1(\pm 1)\Theta_1(\pm 1) = 0, \quad \Theta_1(\pm R) = 0, \quad \Theta_1(\pm 1/R) = 0 \tag{2.33}$$

$$I_2(\pm i) + H_2(\pm i)\chi_2(\pm i)\Theta_2(\pm i) = 0, \quad \Theta_2(\pm iR) = 0, \quad \Theta_2(\pm i/R) = 0 \tag{2.34}$$

Conditions (2.22), (2.25) and (2.29) must be connected with these conditions, and hence we obtain 17 conditions for the parameters b_j $(j=1, \ldots, 8)$, a_k $(k=1, \ldots, 4)$, k_0 and R and for the constants $c_0^{(k)}, c_1^{(k)}$ $(k = 1, \ldots, 4)$, k_1, k_2.

Substituting the quantities (2.32) into Eq. (2.16), we obtain the function $W(\zeta)$, after which, from formula (2.12), we obtain both the conformally mapped function (and thus the analytical form of the unknown part of the boundary)

$$\omega(\zeta) = [p(\kappa-1)]^{-1}W(\zeta), \quad 1 < |\zeta| < R$$

and the function

$$\psi(z) = \psi[\omega(\zeta)] = \psi_0(\zeta) = \overline{W(1/\bar{\zeta})}, \quad 1/R < |\zeta| < 1$$

which, together with the function $\varphi(z)$, determines the bending of the midsurface of the plate (Goursat's formula).

Despite the fact that the equation of the required contour is represented in analytical form, there are considerable mathematical difficulties in determining the form of this contour in practice. This is due, primarily, to the fact that we have 17 conditions for selecting the 24 unknown parameters (even when there is cylindrical symmetry we have 2 conditions for 5 of the parameters). We will therefore consider a special case when there is no contour L_1, and the contour L_0 is a square, on each side of which the same principal normal bending moment M_0 acts (i.e. we will seek an equal strength contour (of an aperture) inside the square).

We will denote by S_0 the quarter of the surface of the plate situated in the first quadrant, and by A_k^0 $(k = 1, \ldots, 5)$ the vertices of its boundary $L^{(0)}$, where $A_5^0 A_1^0$ is part of the required contour (this part is shown hatched in Fig. 2).

Taking into account the results obtained above, we can conclude that $N(t)=0$, $t \in L^{(0)}$, and hence the problem for the whole plate and its part S_0 are equivalent from the mathematical point of view.

Suppose the function $z = \omega^0(\zeta)$ conformally maps the unit semicircle $D_0(|\zeta < 1, Im\, \zeta > 0)$ into the region S_0 and let us use the fact that $a_1^0 = 1$, $a_3^0 = i$, $a_5^0 = -1$ (a_k^0 $(k = 1, \ldots, 5)$) are the originals of the points A_0^k), i.e. the required part of the boundary $L^{(0)}$ is transferred into the section $[-1,1]$.

Proceeding in the same way as above with respect to the functions

$$W_0(\zeta) = \begin{cases} p(\kappa-1)\omega(\zeta), & |\zeta| < 1, \quad \mathrm{Im}\zeta > 0 \\ \overline{\psi_0(\bar{\xi})} + E, & |\zeta| < 1, \quad \mathrm{Im}\zeta < 0 \end{cases} \tag{2.35}$$

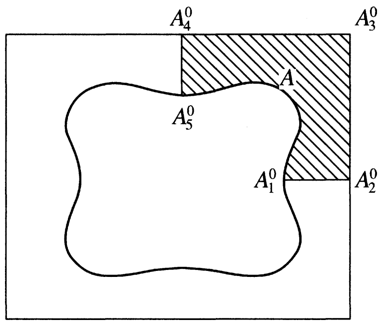

Fig. 2.

(E is an arbitrary (generally speaking, complex) constant), we obtain a Riemann-Hilbert boundary-value problem for the unit circle $D(|\zeta|<1)$, which, in turn, by the method of conformal joining (where the joining function is the Zhukovskii function $\zeta=\zeta+1/\zeta$) can be reduced to a boundary-value problem of linear conjugation for the plane cut along the section [−2,2] of the real axis

$$\begin{aligned} &\Phi^+(t)-\Phi^-(t) = 0, \quad t\in[\delta,2]; \quad \Phi^+(t)+\Phi^-(t) = H_1, \quad t\in[0,\delta] \\ &\Phi^+(t)-\Phi^-(t) = iH_2, \quad t\in[-\delta,0]; \quad \Phi^+(t)+\Phi^-(t) = 0, \quad t\in[-2,-\delta] \end{aligned} \tag{2.36}$$

where

$$\Phi(\xi) = \Psi[\zeta(\xi)]; \quad \Psi(\zeta) = [W_0(\zeta)-iW_0(-\zeta)]/2, \quad \delta = 2\cos\vartheta, \quad (\vartheta = \arg a_2^0)$$

and H_j ($j=1, 2$) are certain real constants. We will seek solutions of the problem of class $h(-2, -\delta, 0, \delta, 2)$.

The necessary and sufficient condition for a problem of this class to be solvable has the form[8]

$$iH_2\int_{-\delta}^{0}\frac{dt}{\chi(t)}+H_1\int_{0}^{\delta}\frac{dt}{\chi(t)} = 0 \tag{2.37}$$

while the solution itself is given by the formula

$$\Phi(\xi) = \frac{\chi(\xi)}{2\pi i}\left[\int_{-\delta}^{0}\frac{iH_2dt}{\chi(t)(t-\xi)}+\int_{0}^{\delta}\frac{H_1dt}{\chi(t)(t-\xi)}\right], \quad \chi(\xi) = \sqrt{(\xi+2)(\xi+\delta)\xi(\xi-\delta)} \tag{2.38}$$

The integrals in formulae (2.37) and (2.38) are elliptic integrals of the first and third kind.

In the first approximation with respect to the parameter $k=\sqrt{2-\delta}/2$, from formula (2.37) we obtain

$$k_0 = \frac{36M_0}{(18-\delta)a}, \quad k_0\in\left(\frac{2M_0}{a}, \frac{9M_0}{4a}\right), \quad 0<\delta<2$$

(a is the length of the side of the square).

In the same way we obtain from formula (2.38)

$$\Phi(\xi) = \frac{\sqrt{\delta}p(\kappa-1)}{16\xi^2}\chi(\xi)$$

and from formula (2.35) the equation of the required contour will have the form

$$\omega_0(\xi) = \omega^0[\zeta(\xi)] = \frac{a\sqrt{\delta}}{16}\sqrt{1-\frac{\delta^2}{\xi^2}}\left(\sqrt{1+\frac{2}{\xi}}+i\sqrt{1-\frac{2}{\xi}}\right), \quad |\xi|\ge 2 \tag{2.39}$$

We similarly obtain

$$\psi_0(\zeta(\xi)) = -\frac{k_0}{4D^0(1-\delta)}\overline{\omega_0(\xi)}, \quad |\xi|\ge 2$$

We will carry out some analysis of the results obtained.

We will denote the abscissae of the points A_1^0, and A (A is the middle of the coutour $A_5^0\,A_1^0$) by x_1 and x^*. We will have from formula (2.39)

$$x_1 = \frac{\sqrt{2}}{32}a\sqrt{\delta}\sqrt{4-\delta^2}, \quad x^* = \frac{\sqrt{\delta}}{16}a$$

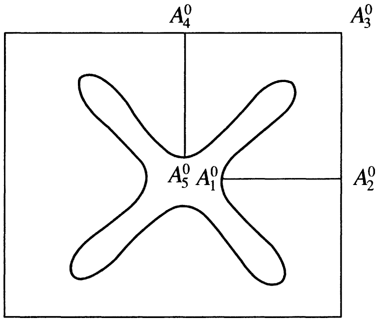

Fig. 3.

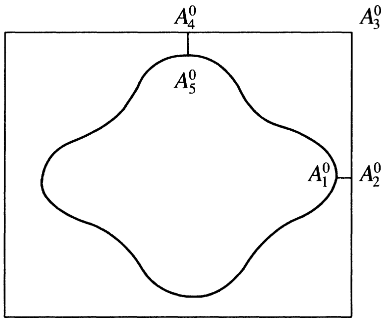

Fig. 4.

Hence we conclude that

$$0<\delta<\sqrt{2},\quad x_1>x^* \quad \text{for} \quad k_0\in\left(\frac{2M_0}{a}, \frac{36M_0}{(18-\sqrt{2})a}\right)$$

$$\delta=\sqrt{2},\quad x_1=x^*=\frac{2}{16}a \quad \text{for} \quad k_0=\frac{36M_0}{(18-\sqrt{2})a}$$

(in the last case the aperture has a shape similar to a hypocycloid (Fig. 2). We also obtain that $x_1\to 0, x^*\to\frac{\sqrt{2}}{16}a$ as $k_0\to\frac{9\,M_0}{4a}$, $\delta\to 0$ (Fig. 3), and when $k_0\to\frac{108\,M_0}{(54-2\sqrt{3})a}$, $\delta=\frac{2\sqrt{3}}{3}$ the coordinate x_1 reaches its maximum, where $x_{1\ \max}=\frac{a}{2^{5/2}3^{3/4}}$, $x^*=\frac{\sqrt{3}}{2}x_{1\ \max}$ (Fig. 4).

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Translated by R.C.G.